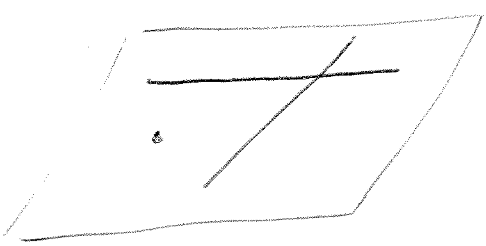
§ Pondering the Projective Plane

Goals
- Define a "geometry" and the Projective Plane
- Give some ways to think about it & why it's worth studying.

"Def": Plane Geometry
- points (underlying set)
- lines (~~subsets~~ collection of subsets)
- axioms (rules)

Ex. Euclidean Plane.

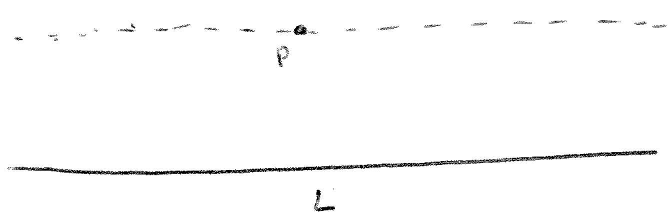


Euclid - "5" axioms

Hilbert - "20"

5^{th} axiom - Parallel Postulate

Given a line L, $p \notin L$, $\exists!$ line through p not intersecting L



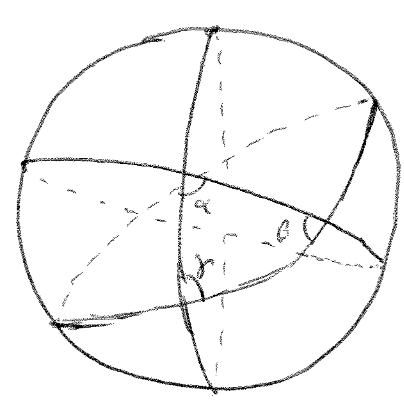
"mysterious" "less constructive"

Q: Can you derive the ~~firs~~ 5^{th} axiom from the other 4?

No - there exist geometries satisfying the first 4, not the 5^{th}

Spherical Geometry

- points - points on a ~~shere~~ sphere
- lines - great circles on a sphere.



Parallel Postulate States

Every pair of lines intersects at 2 points

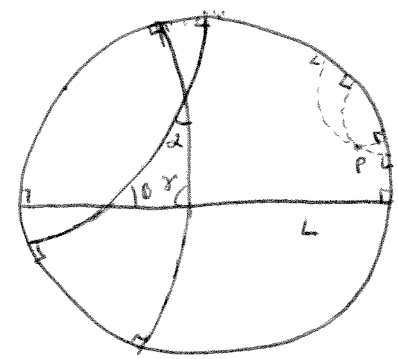
$\Rightarrow$ no parallel lines

Fact: $\text{Area}(\Delta) = \alpha + \beta + \gamma - \pi$

$\Rightarrow$ sum of angles $\geq \pi$

Hyperbolic Geometry (talk to Minsky / Margulis / anyone)

- Points are $\{z \in \mathbb{C} \mid |z| < 1\}$
- Lines are lines + circles intersecting unit circle at $90°$



Parallel Postulate States

Given $p \notin L$, $\exists$ infinitely many "parallel" lines through p.

- draw

Fact: $\text{Area}(\Delta) = \pi - (\alpha + \beta + \gamma)$

$\Rightarrow$ sum of angles $\leq \pi$

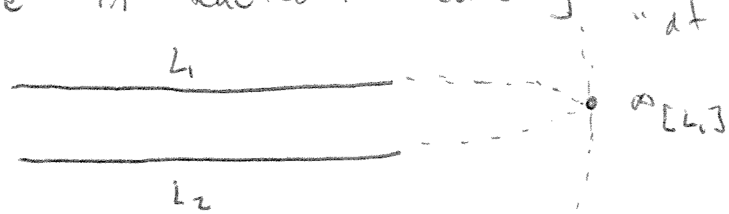
Projective Geometry (talk to Sam Payne / anyone)

"nicest" possible geometry "containing" Euclidean Geometry

Want: Every pair of lines intersects in exactly one point.

(Parallel postulate false)

Failure in Euclidean Geometry "at infinity"

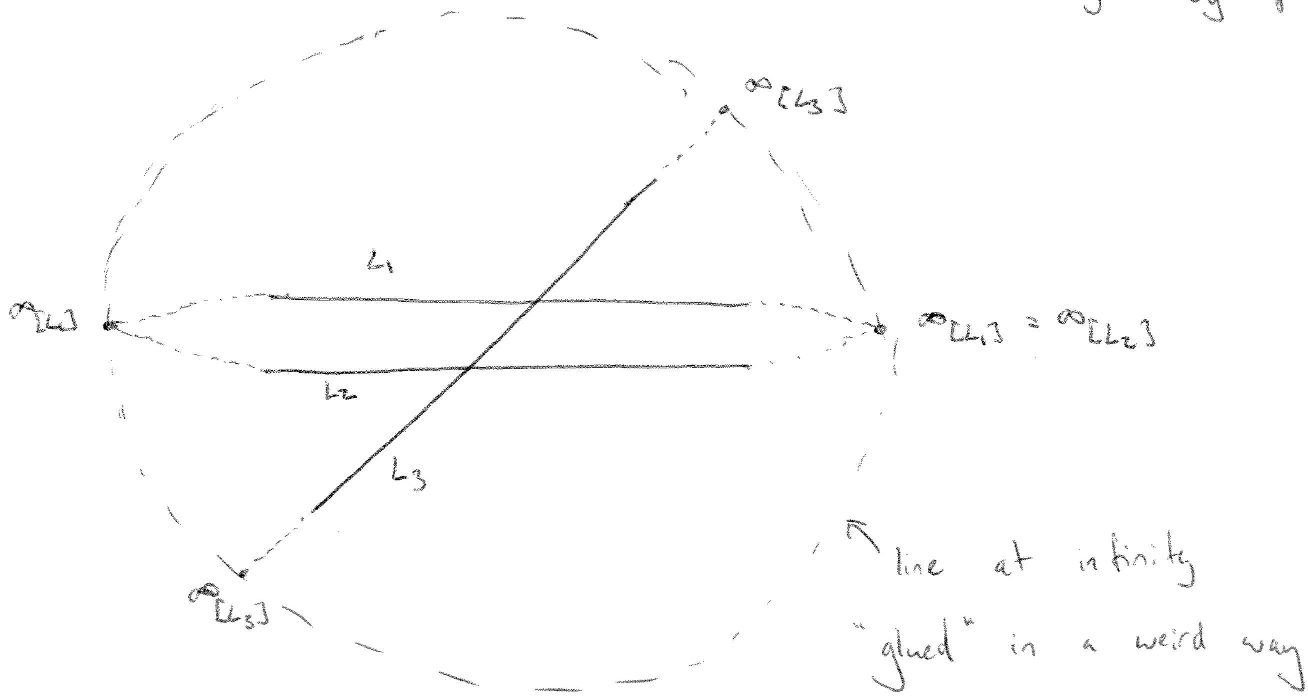


$\infty_{[L_1]}$ - "point at infinity" for lines parallel to L_1

(if $L_1 \parallel L_2$, $\infty_{[L_1]} = \infty_{[L_2]}$

§ Projective Plane

- Points - Euclidean Plane $\cup \{\infty_{[L]} \mid L \text{ a line}\}$.
- Lines - $L \cup \{\infty_{[L]}\}$ - L a Euclidean line.
  - "line at infinity" - $\bigcup_{L \text{ line}} \{\infty_{[L]}\}$ - want there to exist $\exists!$ line through every 2 pts.



Check:
- ~~[illegible]~~ $\exists!$ line through every pair of points
- Every 2 lines intersect at exactly one point.

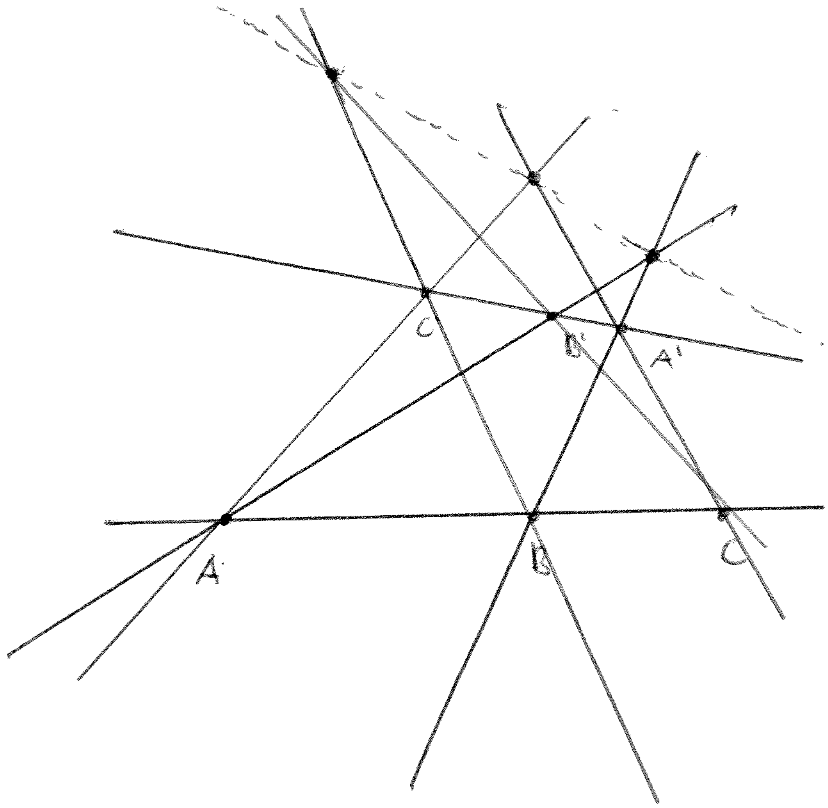
Pappus Thm: (Euclidean)

Let A, B, C collinear, a, b, c collinear. Then

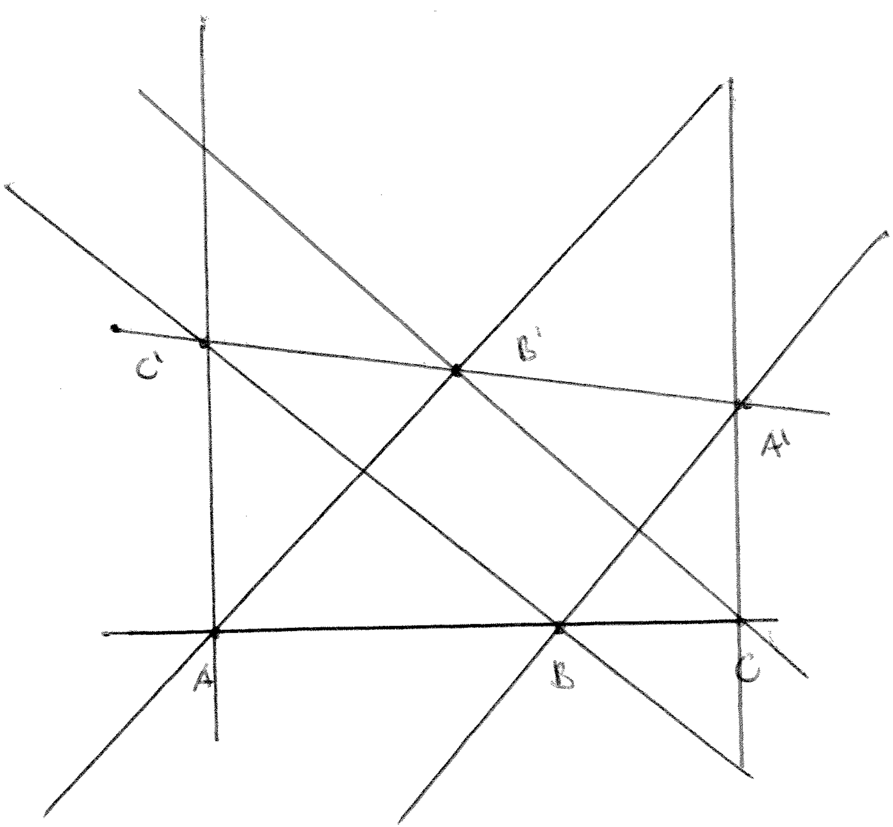
$$\overline{AC'} \cap \overline{CA'}, \quad \overline{AB'} \cap \overline{BA'}, \quad \overline{BC'} \cap \overline{CB'}$$

are collinear unless one pair of lines is parallel in which case bad stuff.

"Generic" "Good Case"



"Bad Case"



Pappus' Thm (Projective)

Let A, B, C be collinear, A', B', C' collinear.

Then $\overline{AB'} \cap \overline{BA'}$, $\overline{AC'} \cap \overline{CA'}$, $\overline{BC'} \cap \overline{CB'}$ are collinear.

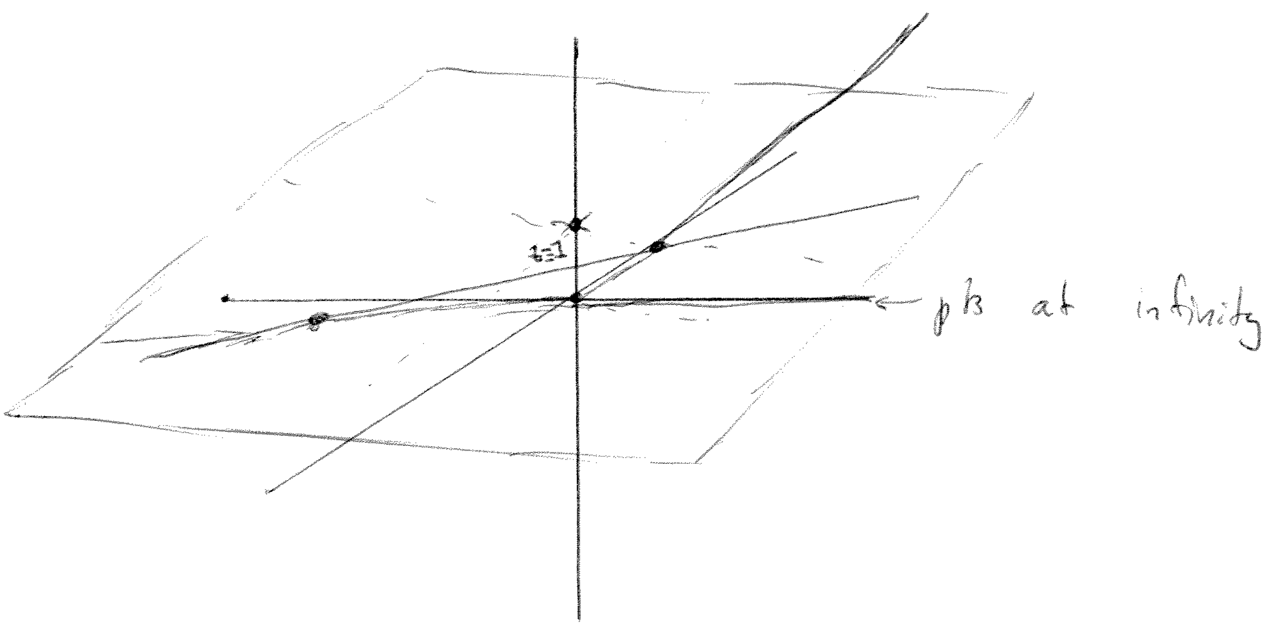
§ Projective "Coordinates"

"Model" projective plane ~~as~~

- Vector space - $\mathbb{R}^3$
  - "points" - 1 dim subspaces. - line through origin.
  - "lines" - 2 dim subspaces - plane through origin.

Check axioms

• $\exists!$ "line" through 2 "pts".	$\longleftrightarrow$	• $\exists!$ 2 dim subspace containing 2 1-dim subspaces
• 2 lines intersect in exactly 1 pt.		• 2 2dim subspaces intersect at a unique 1-dim subspace

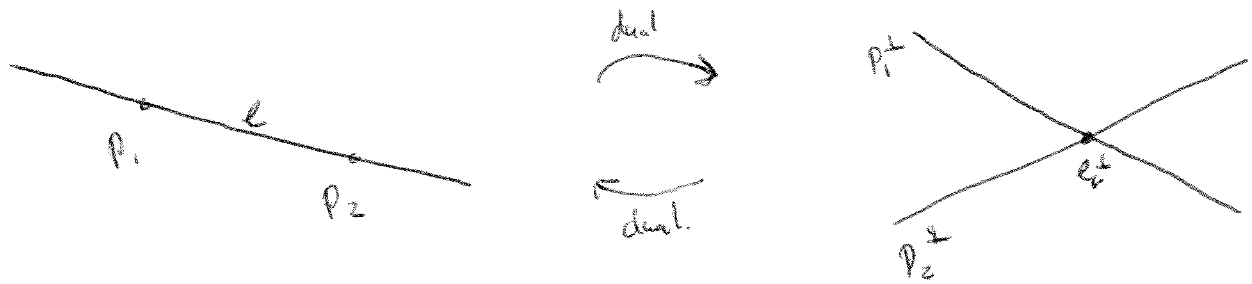


Rmks

- This model generalizes to any field
- Gives you a group of "symmetries"

  3x3 matrices act on $\mathbb{R}^3$ - all points are "the same"

§ Duality : turn "points" into "lines" and preserve incidence structure.



Claim: Orthogonal complement works.

p 1-dim subspace $\rightsquigarrow$ $p^\perp$ 2-dim subspace

ℓ 2-dim subspace $\rightsquigarrow$ $\ell^\perp$ 2-dim subspace.

If I have a theorem, I get a dual Theorem for free!

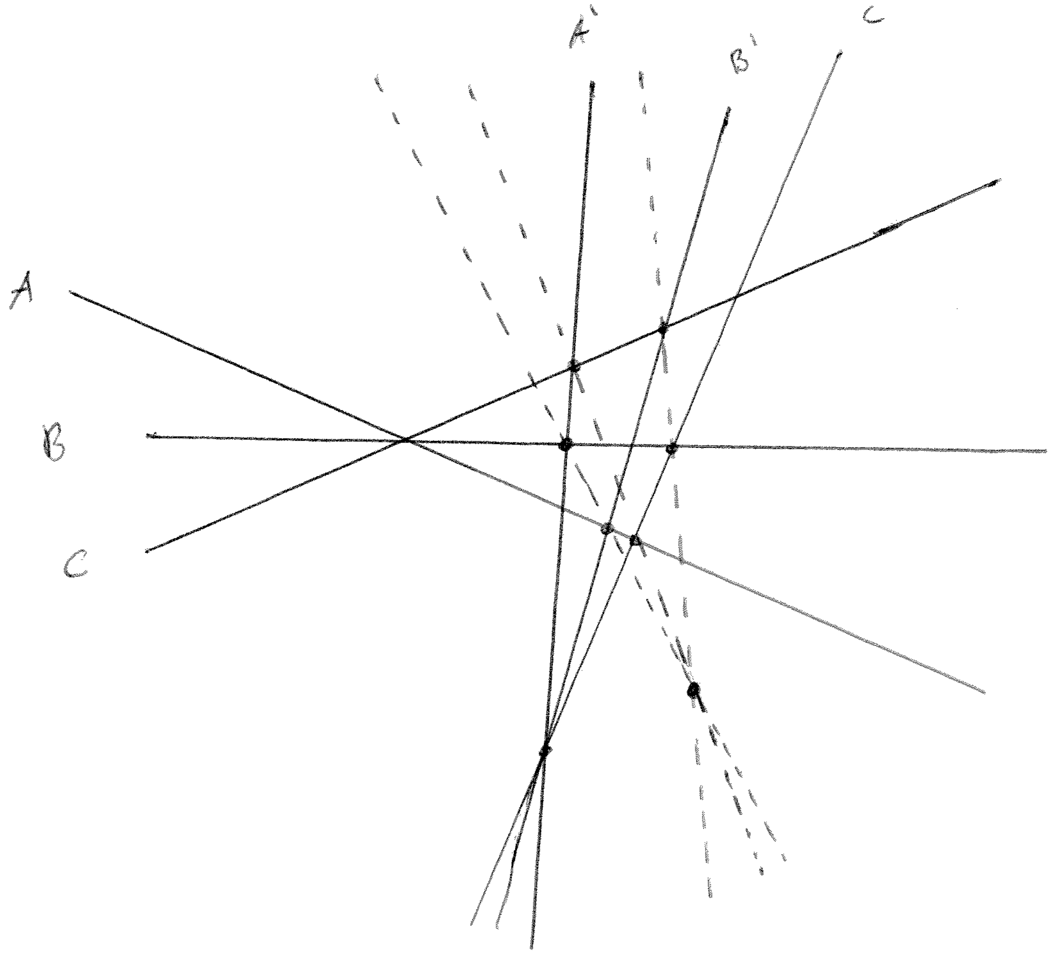
Pappus (Dual) Thm:

If lines A, B, C intersect at a point,
lines A', B', C' intersect at a point

then $\overline{A \cap B', B \cap A'}$ $\overline{A \cap C', C \cap A'}$ $\overline{B \cap C' \ C \cap B'}$

intersect at 1 point.

Pappus' Dual Theorem

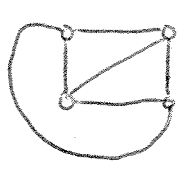


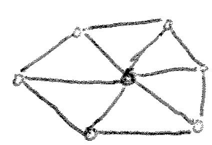
§ "Real" Planar Graphs

Def: Planar graph is a graph you can draw in the Euclidean plane with non-intersecting edges.

Ex.

K_4

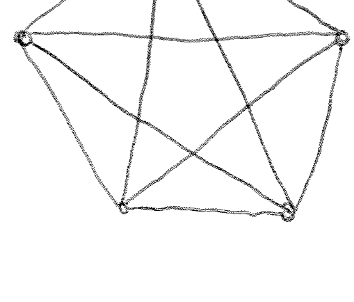




Fact: K_5 is not planar.

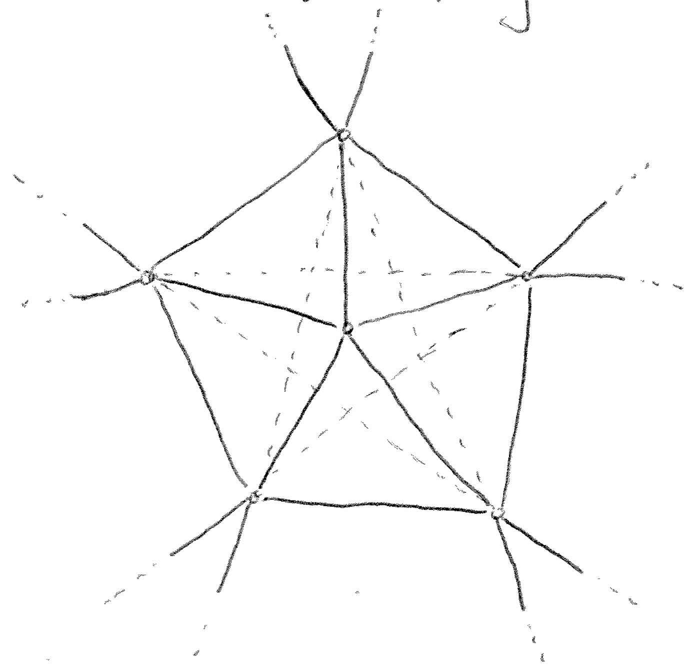
Why? Euler's Formula

$V - E + F = 2$



Q: Is K_5 "planar" in projective plane?

Idea: Use extra lines at infinity to make it planar.

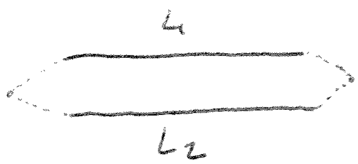


Q1) Is K_6 "real" planar? Yes!

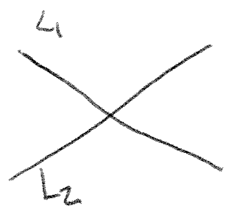
Q2: " " K_7 " " "?

Q3: Why does Euler's formula proof fail?

If time:



- L_1 always above L_2



- should cross at intersection point.

Q: What's going on here?